The Navier-Stokes equations

\# Continuity (G.1 / G.1')

\# Linear momentum $\longrightarrow$ Cauchy's Eqn. (G.2)

\# Constitutive relations $\rightarrow$ Newtonian fluids. (G.3 / G.3')

Consider the simplest model for viscous flows:

Incompressible, homogenous, constant property flows.

G.1 $\rightarrow$ (∵ Incompressible)

G.2 $\rightarrow$ $\tilde{\rho}_0 \frac{D\vec{V}}{Dt} =$ (∵ homogenous)

G.3 $\rightarrow$ $\bar{\bar{\sigma}} =$ (∵ constant viscosity)

For obtaining the Navier-Stokes equations, substitute $\bar{\bar{\sigma}}$ from G.3 into G.2.

$\Rightarrow \nabla \cdot \bar{\bar{\sigma}} =$

Div. of isotropic part Div. of strain rate tensor

Now, let us examine the two terms on the R.H.S of the above equation.

δ_{jk} only contributes when $j = k$

For orthogonal systems

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

$$= -\frac{\partial p}{h_i \partial x_i}\delta_{ij}\hat{e}_j + p\,\delta_{ij}\frac{\partial \hat{e}_j}{h_i \partial x_i} + p\left(\hat{e}_i \cdot \frac{\partial \hat{e}_j}{h_i \partial x_i}\right)\hat{e}_j$$

$$= -\overbrace{\frac{\partial p}{h_i \partial x_i}\hat{e}_i}^{=\nabla p} + p\frac{\partial \hat{e}_i}{h_i \partial x_i} + p\left(\hat{e}_i \cdot \frac{\partial \hat{e}_j}{h_i \partial x_i}\right)\hat{e}_j$$

General expression for pressure force/Vol in orthogonal systems.

For cartesion

$$\nabla \cdot (-p\bar{\bar{I}}) =$$

For cylindrical

For both cartesian and cylindrical

$$\nabla \cdot (-p\bar{\bar{I}}) = -\nabla p$$

The other term involving strain rate tensor can be simplified as,

$$\nabla \cdot \bar{\bar{S}} =$$

general result for any tensor of rank 2.

For Cartesian:

Instead of writing ∂x_i^2 we write $\partial x_i \partial x_i$, since i is repeating index and we need to do summation

$\nabla \cdot \vec{v} = 0$

The order is changed in between ∂x_i & ∂x_k to obtain $\nabla \cdot \vec{v} = 0$. In case of differentiation this can be done

$$\nabla\cdot\bar{\bar{S}} \; [illegible] \; \hat{e}_x = \frac{1}{2}\nabla^2\vec{V}$$

For cylindrical:

$$\nabla\cdot\bar{\bar{S}} = [illegible]$$

[illegible]

[illegible]

+ The above expression has already been derived earlier.

+ Using the strain rate tensor components in cylindrical ~~components~~ coordinates, we can express after manipulation.

$$\nabla\cdot\bar{\bar{S}} = [illegible]$$

[illegible]

[illegible]

The R.H.S is equal to $\nabla^2\vec{V}$

P.S: Do it as H.W

$\nabla\cdot\bar{\bar{S}} = \frac{1}{2}\nabla^2\vec{V}$, This can be verified by applying ∇^2 in cylindrical coords. on $\vec{V}$. Be careful with unit vectors $\hat{e}_r, \hat{e}_\theta$)

Therefore, finally, the linear momentum or G.2 eqn can be expressed for incompressible, homogenous, constant property flow of a Newtonian fluid as,

$$\rho_0 \frac{D\vec{V}}{Dt} = \rho_0 \vec{B}_m - \underbrace{\nabla p + \mu_0 \nabla^2 \vec{V}}_{\nabla \cdot \bar{\bar{\sigma}}}$$

The above equation holds for cartesian as well as cylindrical coordinates.

Finally, we have the following set of equations in primitive variables (velocity, pressure) that govern the viscous, incompressible, homogenous constant property flow of a Newtonian fluid.

viscous force/vol

$$\rho_0 \frac{D\vec{V}}{Dt} = \qquad - \qquad +$$

Pr. force/vol

These are the Navier-Stokes equations

These equations form a complete model as far as no. of variables and no. of equations are concerned.

$$\text{Variables} \equiv (\vec{V}, p) \longrightarrow 04$$

$$\text{Equations} \equiv (\underbrace{01}_{\text{continuity}} + \underbrace{03}_{\text{Momentum}}) \longrightarrow 04$$

Further, the momentum equation reveals that for incomp. homogenous, const. prop. flow of a Newtonian fluid, the viscous force acting on a fluid particle per unit volume $= \mu_0 \nabla^2 \vec{V}$

The linear momentum equation reduces to the Euler's eqn if viscous force is neglected. We already covered Euler's eqn in our earlier course on Fluid Mechanics

Summary

1. The procedure of applying basic laws to a fluid particle directly gives the governing equations for fluid motion.

2. For any type of viscous fluid, the basic governing equations (physical laws):

continuity : $\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\vec{V} = 0$

→ Linear Momentum : $\rho\frac{D\vec{V}}{Dt} =$

→ Constitutive relations

$$\bar{\bar{\sigma}}^{(v)} = func(\bar{\bar{S}}, t)$$

or

$$= func(\bar{\bar{S}})$$

3. For Newtonian fluids

$$\bar{\bar{\sigma}}^{(v)} =$$

$$\lambda =$$ — Stoke's hypothesis

4. The basic physical laws need to be supplemented with equation of states like,

$$\rho = f(p, T) \quad , \quad \mu = f(T, p), \ldots$$

in order to have as many equations as the number of flow variables contained in those equations.

5. The Navier-Stokes equations for viscous, incomp. homogenous, constant property flow for orthogonal coordinate systems (Cartesian, cylindrical, and even spherical) are,

$$\nabla \cdot \vec{V} = 0$$

$$\rho_0 \underbrace{\left[\qquad \qquad \right]}_{\frac{D\vec{V}}{Dt}} =$$

Mathematical properties of N-S equations and Boundary conditions.

Incomp. homog. constt. prop. flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\rho_0\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] =$$

Cylindrical $\rightarrow$ Do it as H.W.

Observations:

1. N-S equations are a set of P.D.E's (4 Nos, for 04 unknowns u, v, w, p)

2. They are coupled equations. We do not have explicit equations for u, v, w and p.

$\Rightarrow$ Solution of each influences the other. Equations have to be solved simultaneously or as a system of equations.

3. The momentum equations have a non-linear term
$\rightarrow$ convective acceleration.

. "Analytical solutions are extremely limited."
Infact much of the mathematical difficulty, complexity in fluid behaviour arises out of this non-linear character.

4. Highest order of derivatives

2nd order in space for velocity components

$$\left(\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 w}{\partial x^2}, \ldots \right)$$

1st order in time for velocity components

$$\left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t} \right)$$

1st order in space for pressure $\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \right)$

Implications on boundary/initial condition requirements:

+ In obtaining particular solutions to ODE

Order of equation = Number of conditions required on the variable

+ A similar requirement exists for P.D.E. A difference is that order of equation is associated with each

Independent variable — space coordinates
— time.

Thus, we can summarize the order of N-S equations for each dependent flow variable u, v, w, p with respect to independent variables (x, y, z, t) and the maximum number of boundary and initial conditions required for obtaining particular solutions.

	space coord.			time
	x	y	z	t
u-vel	Order=2 BC=2	O=2 BC=2	O=2 BC=2	O=1 IC=1
v-vel	"	"	"	"
w-vel	"	"	"	"
p	O=1 BC=1	O=1 BC=1	O=1 BC=1	

5. Alternate formulations

i) Conservative body forces:

Any force whose work is determined by the final displacement of the object acted upon. The work done does not depend upon the path taken.

A body force is conservative if:

$$\nabla \times \vec{B}_m = 0$$

Using vector identity

$\vec{B}_m$ can be expressed as

In the N-S equation, the conservative body forces can be combined with pressure forces,

→ effective or motion pr.

p_m or p_{eff}

Consider the example of gravity which is a conservative body force.

Thus, for flows that are influenced by both gravitational and pressure forces, the effects of both can be combined via a single variable: effective/motion pressure.
The momentum equation for conservative gravity or Body forces becomes:

Pressure + body
force force

For a fluid at rest,

at all points.

This is the reason for calling it as motion pressure, since its gradient exists only for the case of fluid motion.
This approach of combining conservative body forces with pressure is useful when both forces are involved. When the flow is driven by gravity alone, this approach, though valid is not very useful.

(ii) Eliminating "pressure" from N-S equations.

: Velocity - Vorticity formulations:

Consider the following identities in vector calculus,

Using the first and third identity, we have.

The momentum eqn can be expressed as

Taking curl on both sides. this is done in order to eliminate p

$\frac{V^2}{2}$ & p are scalar

(Vorticity Transport eq)

The incompressible condition can also be used to obtain

relation between velocity and vorticity.

This eqn. alongwith Vorticity Transport eqn. constitute a Velocity-Vorticity formulation of an incomp. homogenous, constant property flow.

Note:

1. For 2D flows $\vec{\Omega} \perp \nabla \vec{V}$

$$\Rightarrow \quad \vec{\Omega} \cdot \nabla \vec{V} = 0$$

$\Rightarrow$ Vorticity Transport eq. becomes.

1.

2. For conservative body forces.

Conservative body forces do not generate vorticity